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Counting the microstates of a vacuum black ring

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Abstract: The Bekenstein-Hawking entropy of an extremal vacuum black ring is derived from a microscopic counting of states. The entropy of extremal Kaluza-Klein black holes with ergospheres is also derived.

Keywords: Black Holes in String Theory, Black Holes.

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Contents

1. Introduction

There has been recent progress in using string theory to provide a microscopic calculation of the entropy of extremal vacuum black holes [1 − 3]. In particular, Horowitz and Roberts have shown how the Bekenstein-Hawking entropy of an extremal Kerr black hole

$$S = 2\pi|J| \tag{1.1}$$

can been reproduced from a statistical counting of microstates [3]. Extremality is important in these calculations since extremal black holes obey an attractor mechanism (see [4] for a review), even when rotating [5], which explains why the entropy of such black holes does not change as the string coupling is decreased. This implies that the entropy calculated from a solution of classical gravity can be compared directly with the entropy calculated microscopically [6].

In this paper, I shall extend these calculations to a different class of vacuum black holes: black rings. Black rings with a single non-vanishing angular momentum were constructed in [7] but they do not admit a regular extremal limit. However, black rings with two angular momenta, constructed in [8], do.[1] An extremal vacuum black ring has two parameters: the two angular momenta J_1, J_2. The present paper is motivated by the observation that the Bekenstein-Hawking entropy of an extremal vacuum black ring is

$$S = 2\pi|J_2|. \tag{1.2}$$

[1]Some physical properties of the solutions of [8] have been discussed in [9].

The similarity with equation (1.1) suggests that a microscopic derivation of this result may be possible.

The similarity of equations (1.1) and (1.2) arises from the fact that an extremal black ring has a near-horizon geometry that is isometric to the near-horizon geometry of an extremal boosted Kerr string with $J = J_2$ [10]. The entropy of the latter is independent of the boost, and hence equals the entropy of an unboosted Kerr string. Upon dimensional reduction, this is just the entropy of an extremal Kerr black hole.

The idea that we shall exploit in this paper arises from the study of BPS black rings [11], for which succesful microscopic calculations of the entropy have been performed [12, 13]. Since black rings can be regarded as rotating loops of black string, these calculations start by assuming that the low-energy dynamics of a BPS black ring should be described by the CFT that governs the low energy dynamics of the corresponding BPS black string. More precisely, the calculations involve identifying the charges of a BPS black ring with the charges of a BPS boosted black string wrapped on a Kaluza-Klein circle, and then calculating the microscopic entropy of the latter. One might expect this approach to work for "skinny rings", for which the radius R_1 of the S^1 (of the $S^1 \times S^2$ horizon) is much greater than the radius R_2 of the S^2. Indeed, for extremal dipole rings one obtains the correct result for the entropy calculated this way for large R_1/R_2 [14]. For BPS rings, it turns out that this microscopic calculation correctly reproduces the Bekenstein-Hawking entropy for *arbitrary* R_1/R_2 [12, 13]. (In fact, the entropy of extremal dipole rings can also be calculated for arbitrary R_1/R_2 [15].)

For extremal vacuum rings, we shall see that $R_1/R_2 \sim J_1/J_2$. Hence we might expect the above method to work for large J_1/J_2. However, the entropy (1.2) is independent of J_1. Hence it is independent of R_1/R_2. Phrasing things differently, the leading term in the expansion of the entropy in large R_1/R_2 (i.e. large J_1/J_2) is exact. This is an encouraging sign that a microscopic state counting based on regarding the black ring as a boosted black string, which works so well for BPS rings, may also work for extremal vacuum rings with arbitrary J_1/J_2.

The idea, then, is to take the microscopic theory of the black ring to be the theory governing an extremal boosted Kerr black string. This is the theory used for the Kerr microstate counting in [3]. We need to map the charges J_1, J_2 of the black ring to the black string charges. For BPS black rings, there is disagreement over how to do this, with two different methods proposed [12, 13]. However, in the vacuum case studied here, it seems quite clear cut: the isometry between the black ring and black string near-horizon geometries fixes the identification uniquely. In any case, the only result we need to do the calculation is the identification of the black string angular momentum J with the black ring angular momentum J_2, which looks uncontroversial.

Our microscopic calculation, which is a slight modification of [3] also allows us to extend the results of [2] governing "ergo-branch" Kaluza-Klein black holes to arbitrarily high angular momentum.

This paper is organized as follows. In section 2, we present some properties of extremal vacuum black rings. In section 3, we use the isometry between near-horizon geometries to determine the charges of the boosted black string that we will use in the entropy calculation.

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Section 4 contains the entropy calculation. Section 5 contains a brief discussion.

2. Extremal black ring

An extremal vacuum black ring is specified by two parameters $k > 0$ and $0 < \lambda < 2$. k has dimensions of length and sets a scale for the solution. λ is dimensionless. The mass is

$$M = \frac{12k^2\pi\lambda}{G_5(2-\lambda)^2}, \tag{2.1}$$

and the angular momenta are (choosing them to be positive)

$$J_1 = \frac{8k^3\pi\lambda(4+8\lambda+\lambda^2)}{G_5(2-\lambda)^3(2+\lambda)}, \qquad J_2 = \frac{32k^3\pi\lambda^2}{G_5(2-\lambda)^3(2+\lambda)}. \tag{2.2}$$

The solution is uniquely determined by its conserved charges, in constrast with non-extremal rings. To see this, note that

$$\frac{J_2}{J_1} = \frac{4\lambda}{4+8\lambda+\lambda^2}. \tag{2.3}$$

The function on the r.h.s. is monotonically increasing for $0 < \lambda < 2$. Hence λ is uniquely determined by J_2/J_1, and we have $0 < J_2/J_1 < 1/3$, i.e.,

$$J_1 > 3J_2 > 0. \tag{2.4}$$

Having fixed λ, k is uniquely specified by the value of, say, J_1. Hence the solution is uniquely specified by (J_1, J_2) in the range (2.4). In the limit $J_1/J_2 \to 3$, the solution probably becomes an extremal Myers-Perry [16] solution.[2] Eliminating k and λ from M gives

$$M^3 = \frac{27\pi}{4G_5} J_2(J_1 - J_2). \tag{2.5}$$

Equation (2.4) implies that

$$\frac{27\pi}{2G_5} J_2^2 < M^3 < \frac{3\pi}{2G_5} J_1^2. \tag{2.6}$$

At the horizon, the radius of the S^1 varies over the S^2. At the poles of the S^2, the S^1 has radius

$$R_1 = \frac{2k(2+\lambda)}{2-\lambda}. \tag{2.7}$$

(At the equator, the radius of the S^1 is $\sqrt{3/2}$ times larger.) This can be rewritten as

$$\frac{R_1^3}{G_5} = \frac{4(J_1 - J_2)^2}{\pi J_2}. \tag{2.8}$$

[2] Although I have not checked this. Evidence in favour of this comes from comparing the mass of an extremal ring to the mass of an extremal MP solution with the same angular momenta: $M_{\mathrm{MP}}^3 = 27\pi(J_1+J_2)^3/(32G_5)$. One finds that $M_{\mathrm{ring}}/M_{\mathrm{MP}}$ is a monotonic increasing function of J_2/J_1, attaining its maximum value of 1 as $J_2/J_1 \to 1/3$. Hence the masses of the solutions agree in this limit. The ratio of the entropies is $S_{\mathrm{ring}}/S_{\mathrm{MP}} = (J_2/J_1)^{1/2} < 1/\sqrt{3}$, so entropy would be discontinuous in the limit in which the ring became a MP solution, just as happens in the limit in which a BPS black ring approaches a topologically spherical black hole.

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The S^2 is not homogeneous, but we can define an effective radius R_2 by saying that it has area $4\pi R_2^2$. This gives

$$R_2 = \frac{4k\lambda}{4-\lambda^2}, \tag{2.9}$$

which implies

$$\frac{R_2^3}{G_5} = \frac{J_2^2}{2\pi(J_1 - J_2)}, \tag{2.10}$$

Note that

$$\frac{R_1}{R_2} = \frac{2(J_1 - J_2)}{J_2} \tag{2.11}$$

hence extremal rings with $J_1 \gg J_2$ are skinny whereas extremal rings with $J_1 \sim 3J_2$ are fatter with $R_1/R_2 \sim 4$.

In order to neglect higher-derivative corrections, we need R_1 and R_2 to be large in Planck units, which requires $J_2^2 \gg J_1 - J_2 \gg \sqrt{J_2} \gg 1$.

3. Matching to a Kerr string

Take the product of the 4d Kerr solution with a flat direction, boost in this direction, compactify this direction into a circle and then take the extremal limit. This gives the 3-parameter extremal boosted Kerr black string solution. The 3 parameters are the angular momentum J, the number N_0 of units of momentum around the KK circle, and the asymptotic radius R of this circle.

The near-horizon geometry of an extremal vacuum black ring was obtained in [10]. It was shown that this is globally isometric to the near-horizon geometry of an extremal boosted Kerr black string. In order the make the correspondence between black ring and black string precise, we need to relate the 3 parameters of the black string to the 2 parameters of the black ring. The desired relation follows from the isometry between the near-horizon geometries. One finds that

$$J = J_2, \tag{3.1}$$
$$N_0 = J_1 - J_2, \tag{3.2}$$

and $R = R_1/\sqrt{2}$, so

$$\frac{R^3}{G_5} = \frac{\sqrt{2}(J_1 - J_2)^2}{\pi J_2}. \tag{3.3}$$

We should note that, for BPS rings, there is disagreement in the literature over how the parameters of the black string should be related to those of the black ring [12, 13]. In particular, there is disagreement over the value of N_0 for BPS rings. For vacuum rings, the above argument seems clear cut (and appears to favour the proposal of [12] over that of [13]) but it doesn't generalize to BPS rings since the near-horizon solution of the latter contains fewer parameters than the full solution hence matching near-horizon solutions does not allow one to match uniquely parameters in the full solution. Even if one disagrees with the above value for N_0, the argument below is independent of the precise value of N_0 (because the entropy doesn't depend on N_0).

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4. Entropy calculations

4.1 Kaluza-Klein black holes

The boosted Kerr black string can be dimensionally reduced to give an extremal 4d Kaluza-Klein black hole. This solution is specified by its electric charge N_0, which is just the number of units of momentum around the KK circle, and by its angular momentum J. Taking the product of this solution with a 6-torus and interpreting the KK circle as the M-theory circle, this solution carries D0-brane charge N_0. More general extremal KK black holes [17] are parameterized by (N_0, N_6, J) where N_6 is KK monopole charge in 11 dimensions, or equivalently D6-brane charge in 10 dimensions. Such black holes fall into two classes. In the terminology of [5], the "ergo-free branch" of black holes has $J^2 < N_0^2 N_6^2/4$ and the entropy

$$S_{\text{ergo-free}} = 2\pi\sqrt{N_0^2 N_6^2/4 - J^2} \tag{4.1}$$

of such black holes was calculated in [1] by dualizing to a non-BPS 4-charge configuration and arguing that results derived in the BPS case could be extended to this case. "Ergo-branch" black holes have $J^2 > N_0^2 N_6^2/4$ and the entropy

$$S_{\text{ergo}} = 2\pi\sqrt{J^2 - N_0^2 N_6^2/4} \tag{4.2}$$

of such black holes was calculated in [2] assuming that

$$1 - N_0^2 N_6^2/(4J^2) \ll 1. \tag{4.3}$$

This condition arises from requiring that the dual 4-charge configuration admit an AdS_3 factor in its decoupling limit, so that CFT arguments are legitimate. The Kerr string has $N_6 = 0$ so it is on the ergo-branch but does not satisfy the condition (4.3). This problem was circumvented in [3] by an ingenious transformation that interchanges the ergo and ergo-free branches of solutions.

4.2 The method of Horowitz and Roberts

The argument of [3] involved two novel steps that we shall exploit below.

Covering spaces. Consider an extremal KK black hole with parameters (N_0, N_6, J), KK circle radius R and entropy S. Assume $N_6 > 0$, so the KK circle is non-trivally fibered over the 4d spacetime. The topology of the horizon (or spatial infinity) is S^3/Z_{N_6}. If K divides N_6 then one can pass to a K-fold covering space of this solution, keeping the local geometry fixed in 11d Planck units. The new parameters are $(N_0 K^2, N_6/K, KJ)$ and the KK circle has radius KR [3]. Working in the covering space amounts to considering K copies of the original black hole, so the entropy becomes KS.

Branch exchange. Consider a KK black hole with parameters $(N_0, 1, J)$, KK circle radius R and entropy S. Let $R \to \infty$. This gives an asymptotically flat[3] extremal Myers-Perry [16] black hole [2]. The angular momenta in orthogonal planes are $J_{1,2} = N_0/2 \pm J$ [2]. Now perform a reflection to change the sign of J_2. This has the effect of interchanging

[3] If $N_6 > 1$ then the solution would not be asymptotically flat.

N_0 and $2J$. Finally, extrapolate back to finite R. The new solutions has parameters $(2J, 1, N_0/2)$. Hence if the original solution was on the ergo branch then the new solution is on the ergo-free branch and vice-versa. The attractor mechanism ensures that the entropy does not change as R is varied, and a reflection clearly does not change the entropy. Hence the final solution must have the same entropy as the initial solution, as can be checked using (4.1) and (4.2).[4] However, there is no reason for the mass to be invariant and indeed it is not (explicit expressions for the mass are given in [2]).

4.3 Black ring

We have argued above that calculating the entropy of an extremal vacuum black ring should be equivalent to calculating the entropy of an extremal boosted Kerr string. Hence our starting point is the extremal boosted Kerr string, or extremal KK black hole, with parameters $(N_0, 0, J)$. Let S denote the entropy of this solution. T-dualizing this on the entire T^6 gives a solution with parameters $(0, N_0, J)$. The KK circle is now non-trivially fibered over the 4d spacetime with charge N_0. We now take a N_0-fold covering space of this solution whilst keeping the local geometry fixed in Planck units. This amounts to considering N_0 copies of our original black hole. The resulting solution has parameters $(0, 1, N_0 J)$. The radius R of the KK circle increases to $N_0 R$ and the entropy is $N_0 S$.

Next we apply the branch-exchange transformation to obtain a KK black hole with charges $(2N_0 J, 1, 0)$ and entropy $N_0 S$. Note that this is on the ergo-free branch.

Now we T-dualize on T^6 to obtain a KK black hole with charges $(1, 2N_0 J, 0)$ and then take a K-fold cover of the KK circle (where K divides $2N_0 J$), keeping the local geometry fixed in Planck units. This gives a new black hole with parameters $(K^2, 2N_0 J/K, 0)$ and entropy $KN_0 S$.

In summary, we have explained why the entropy of our black hole should be $1/(KN_0)$ times that of an extremal KK black hole with parameters $(K^2, 2N_0 J/K, 0)$. For large $K, N_0 J/K$, the entropy $2\pi K N_0 J$ of the latter was reproduced by a statistical counting of states in [1].[5] Hence this counting predicts an entropy $2\pi J$ for our original black ring. Setting $J = J_2$, this agrees with the Bekenstein-Hawking entropy (1.2).

4.4 General ergo-branch black holes

With slight modification, the above argument can also be applied to general extremal ergo-branch KK black holes in order to relax the condition (4.3). Above we started with $N_6 = 0$ but now we consider an ergo-branch solution with parameters (N_0, N_6, J) and $N_6 > 0$. Let S denote the entropy.

First we go to a N_6-fold covering space of the KK circle keeping the local geometry fixed in Planck units. This gives us a KK black hole with parameters $(N_0 N_6^2, 1, N_6 J)$ and

[4] Note that the point of this argument is that is explains why the solutions with charges $(N_0, 1, J)$ and $(2J, 1, N_0/2)$ have the same entropy, which would otherwise be a mysterious coincidence.

[5] This counting requires that $K^2 = 4k^3 N$ and $2N_0 J/K = 4l^3 N$ for integers k, l, N with $N \gg 1$. We can arrange this e.g. by taking $K = 2n$, $k = 1$, $N = n^2$, $N_0 = l^3$, $J = 4n^3$. Hence, for the ring, $J_1 = l^3 + 4n^3$, $J_2 = 4n^3$. This is a restriction on the charges of the original solution. Similar restrictions apply to [1–3].

entropy $N_6 S$. Now perform a branch-exchange transformation. This gives an ergo-free extremal KK black hole with parameters $(2N_6 J, 1, N_0 N_6^2/2)$ and entropy $N_6 S$.

Next, T-dualize on T^6 to obtain a solution with parameters $(1, 2N_6 J, N_0 N_6^2/2)$, and taking a K-fold cover of the KK circle gives a solution with parameters $(K^2, 2N_6 J/K, K N_0 N_6^2/2)$ and entropy $K N_6 S$. This is a solution whose entropy was calculated microscopically in [1][6] with the result $2\pi K N_6 \sqrt{J^2 - N_0^2 N_6^2/4}$, so dividing by $K N_6$ exactly reproduces the Bekenstein-Hawking entropy (4.2) of our original black hole.

5. Discussion

In this paper, we have presented a microscopic calculation of the entropy of extremal vacuum black rings. Our approach was based on the mapping from a black ring to a black string that has been succesful for BPS black rings [12, 13]. For BPS rings, this approach has several limitations, which were discussed in [18]. Similar limitations apply for vacuum rings. For example, since this approach cannot distinguish a black ring from a boosted black string, it provides no understanding of the lower bound (2.4) on J_1. Furthermore, since the calculation applies only to black rings, and not to asymptotically flat Myers-Perry black holes, it provides no hint of what distinguishes a black ring from a topologically spherical black hole at the microscopic level.

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